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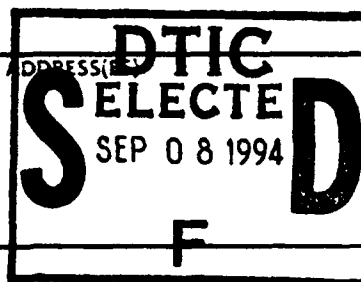
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The nonlinear version of Lorentz Reciprocity cannot be articulated in terms of frequency domain concepts. Similarly, the fact that a Fourier Transform relates an antenna's far field to its sources cannot be used to "explain" why electromagnetic bullets or missiles cannot exist since, by construction, there is no far field for a bullet. Over the last year it has become clear that one has to deal with Maxwell's Equations as a system of hyperbolic p.d.e.'s and avoid the temptation of using elliptic theory which is applicable when taking a Fourier Transform (as engineers are trained to do) and playing with Helmholtz's equation. The way to achieve these goals is to reexamine the "raison d'etre" for the use of the Radon transform in these hyperbolic problems.

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**Geometrical and Topological Methods in
Time Domain Antenna Synthesis**

Prepared for
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1 Introduction and a reorientation

Contract F49620-92-J-0056 dealt with mathematical aspects of "electromagnetic bullets." The research had two objectives:

1a) Given a solenoidal vector field J defined throughout a region R , find, if possible, a set of Clebsch Potentials[27] ξ, η such that

$$J = \text{grad } \xi \times \text{grad } \eta \quad \text{in } R \quad (1)$$

Note: If we pick a point p in R then Clebsch potentials are known to exist in some neighborhood U of p .

1b) In general, find the topological obstructions to finding a set of globally defined Clebsch potentials.

2) Investigate the (time domain) techniques required to formulate a nonlinear version of the Lorentz Reciprocity Law.

From the outset, I mention that much folklore appearing in the published literature has serious errors which must be articulated in order to achieve the objectives of the proposed research. Specifically,

1. It is widely believed (Kuznetsov and Mikhailov[54] are often cited as a source) that a set of global Clebsch potentials can be found by "pulling back" the area formed on the sphere S^2 . In the physics literature this is called an n-field representation. Although the underlying intuition can be obtained by appealing to the gyrovector in micromagnetics[50, 51], I have shown[53] that the gyrovector construction is not possible for a general solenoidal vector field. (The present research follows up on this.)
2. The nonlinear version of Lorentz Reciprocity cannot be articulated in terms of frequency domain concepts. Similarly, the fact that a Fourier Transform relates an antenna's far field to its sources cannot be used to "explain" why electromagnetic bullets or missiles cannot exist since, by construction, there is no far field for a bullet. Over the last year it has become clear that one has to deal with Maxwell's Equations as a system of hyperbolic p.d.e.'s and avoid the temptation of using elliptic theory which is applicable when taking a Fourier Transform (as engineers are trained to do) and playing with Helmholtz's equation. The way to achieve these goals is to reexamine the "raison d'etre" for the use of the Radon transform in these hyperbolic problems.

In reference to this second point, I have discovered that the Mathematics Literature has a natural reason for the appearance of the Radon transform[74, 5]. Specifically, if one tries to describe the lacunas (Latin for hole) of fundamental solutions of hyperbolic p.d.e.'s, one notices that the support (in space-time) of the singular part of a solution lies on a light cone made of rays. These rays are identified with the points of a projective space which form the domain of integration in the Radon transform. Although not part of the original research

plan, pursuit of this observation seems crucial for a deeper understanding of the bullet idea and of reciprocity.

In addition to the ties with lacunas of hyperbolic equations, my research has focused primarily on two areas:

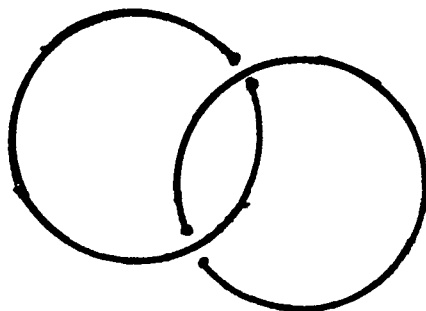
1. The role of higher order linking of flux lines as an obstruction to Clebsch potentials. (i.e. Massey products in a suitable cohomology ring come to play.)
2. Finding an underlying incredible Hamiltonian system with enough structure to indicate obstructions to globally defined Clebsch potentials. (So far we have three such systems but the clearly articulated obstructions are still forthcoming.)

The remainder of the report develops the two above topics with emphasis on the first.

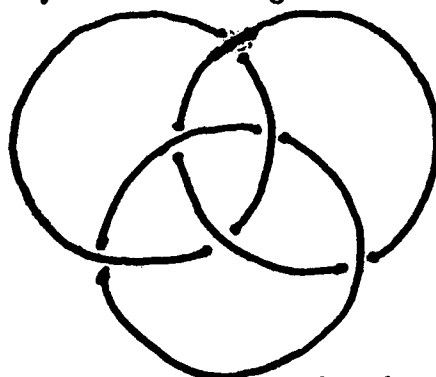
2 Massey Products and obstructions to Clebsch potentials

2.1 Relationship to recent developments in algebraic topology

Massey Products[56, 57, 80] are a way of articulating higher order linking. For example, the curves



are unseparable because they are linked. Higher order linking can be illustrated by Borromean rings:



where the three rings cannot be separated even though pairwise they can be separated. (In a previous publication[48] I have stumbled across a need for Massey products in another context. This higher order linking is important in the context of Clebsch potentials since they describe an obstruction to having a set of single valued globally defined potentials. To develop this connection further, recall that in a neighborhood U of a point p , a solenoidal

vector field J is expressible in terms of Clebsch potentials and hence, it's vector potential (which we denote by F) has a Monge potential representation:

$$\begin{aligned} J &= \text{curl } F & \text{where} \\ F &= \xi \text{ grad } \eta + \text{grad } \psi \text{ in } U \end{aligned} \quad (2)$$

Ties between Clebsch potentials and Massey products are best motivated by appealing to the "helicity of the vector potential." The helicity of a vector field G is given by

$$\int G \cdot \text{curl } G \, dV \quad (3)$$

and its interpretation in terms of twisted and tangled flux lines has been developed by Moffatt in the context of magnetohydrodynamics[63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 2]. Berger and Field[9] have continued this investigation and Berger has given explicit expressions for Massey Products in the notation of pedestrian vector analysis[10]. In my publication[53] I show that an obstruction to globally defined Clebsch potentials follows from the fact that the helicity of any vector potential described by global Monge potentials vanishes. More concretely, nonzero total helicity implies an obstruction.

Further insight into the tie between helicity and the global obstructions to Clebsch potentials can be found by appealing to the Frobenius integrability condition:

$$\left. \begin{aligned} \text{If } F \cdot \text{curl } F &= 0 & \text{in } U \\ \text{then } F &= \xi \text{ grad } \eta & \text{for some } \xi \text{ and } \eta \end{aligned} \right\} \quad (4)$$

A quick calculation shows that the converse of the implication (4) is true. By the gauge freedom of ψ , apparent in (2), we see that setting ψ equal to zero ensures that the helicity of F can be made to vanish in some neighborhood of any point p ! Since the helicity integral (3) subject to boundary constraints is usually a fixed nonzero number we see that the obstruction to globally defined Monge (and hence Clebsch) potentials follows from the twisting and tangling of flux lines and from Massey products describing "linking" of closed flux lines.

The tie between helicity and Massey products or other obstructions to globally defined Clebsch potentials is still incomplete but recent mathematical developments point to a deep connection between these ideas. To see this we note that all of the above arguments can be rephrased in terms of differential forms in order to demonstrate that they are independent of the Riemannian metric on the space. This coordinate free approach then ties into some basic facts and some recent history:

1. There is a non-abelian generalization of the helicity: the "Chern-Simmons Secondary Characteristic Classes,"[26, 73]. That is, if one thinks of a 1-form as a $U(1)$ connection and it's exterior derivative as a curvature, the helicity reduces to a Chern-Simmons form. More generally, if \mathcal{A} is a Lie algebra valued 1-form, and

$$d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \quad (5)$$

the associated curvature, the Chern-Simmons form looks like:

$$\int \text{tr}[\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}] \quad (6)$$

Ed Witten[93] has constructed an associated topological quantum field theory which yielded the Jones polynomial of knot theory[44] and an extension to three-manifolds (as well as a fields medal for Witten). The underlying philosophy is that if one starts with a metric invariant Lagrangian (the Chern Simons form in this case) then the expectation values associated with any metric independent observable are necessarily topological invariants.

2. To most mathematicians, Witten's use of Feynmann path integrals is tantamount to witchcraft, and so an independent approach to the Jones polynomial is sought via orthodox algebraic topology. Such an understanding is beginning to emerge through the work of Vassiliev[89, 12] who uses the Weierstrauss approximation theorem to build a sequence of spaces approximating, and in some sense converging, to the "Space of embeddings of S^1 into \mathbb{R}^3 " (i.e. the space of all knots). Vassiliev's invariants are then extracted from a spectral sequence involving the reduced homology groups of these spaces. Birman and Lin have extracted the Jones polynomial from Vassiliev's invariants. See Birman[12] for a recent survey article (especially Section Four). Finally, it is important to say that the entire formalism goes over to the case of links (i.e. embeddings of several copies of S^1 in \mathbb{R}^3).
3. What is most exciting is the emerging relationship between perturbative approaches to the Chern-Simmons topological quantum field theory and Vassiliev's invariants. The recent work of Baez[6] and Kontsevich[47], appear to be the first successes at making the Feynmann integral approach to Vassiliev invariants a part of orthodox algebraic topology. The bridge to link invariants such as Massey products is also emerging through attempts to decompose Vassiliev's invariants into a sum containing well known invariants and "other terms." The Milnor μ -invariants are known to arise in this way[7, 13]. In the context of knot complements, earlier work[75, 88] has shown that these μ -invariants are just Massey products.
4. Another framework for Jones-like invariants comes from the study of "exactly solvable problems in statistical mechanics," R-matrices, and quantum groups. Although it is beyond my competence to describe these developments, it is important to point out one connection with my previous work on the helicity functional. In[49] I consider the finite element discretization of the helicity functional (3) by means of Whitney forms[90]. Over a tetrahedron Δ , the discretized expression involves six independent edge variables h_{ij} , $1 \leq i, j \leq 4$, where the indices correspond to vertices of the tetrahedron and $h_{ij} = -h_{ji}$. Upon discretization we have

$$\int_{\Delta} \mathbf{H} \cdot \text{curl } \mathbf{H} \, dV \longrightarrow h_{12}h_{34} - h_{13}h_{24} + h_{14}h_{23} \quad (7)$$

The right hand side is the Pfaffian of the skew-symmetric 4 by 4 matrix formed by the h_{ij} and it clearly preserves the metric independence of helicity. *This expression has the same symmetries as the "6-j symbols" which appear in the tensor products of group representations and hence seem to point to a more direct link with quantum group invariants. Although intriguing, a more articulate description of what is going on is not apparent to me or prominent experts whom I have approached.*

The above four points are summarized schematically in Figure 1. Our objective is to articulate how “linking and tangling of flux lines” provide a global obstruction to defining a set of global Clebsch potentials. Noting that Monge potentials can always be chosen so that the helicity density vanishes locally, the helicity integral provides an obstruction to globally defined Clebsch potentials and a bridge to an algebraic description of the obstruction (e.g. Massey products).

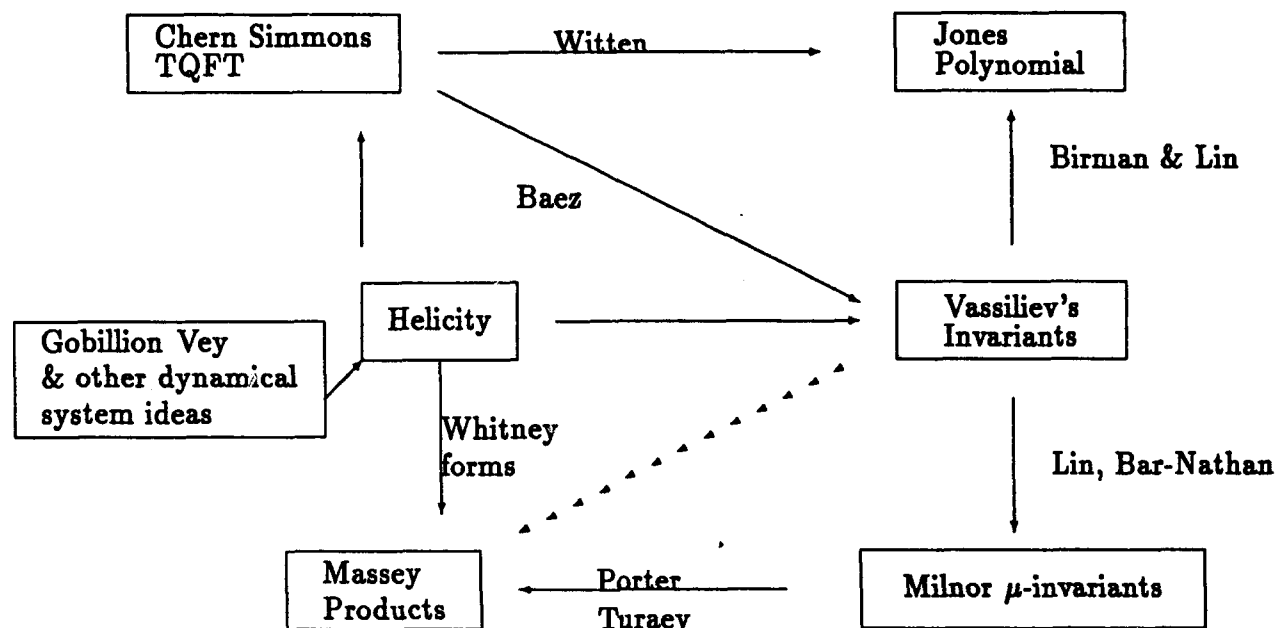


Figure 1: Question: Can Whitney forms and finite elements provide a more direct link?
Possible Answer: Pfaffian Expression for Helicity

2.2 A diversion on Massey products and Morse inequalities

It is my belief that Massey products have a fundamental role to play in critical point theory via the “Mountain Pass Lemma” and its generalizations[81]. The basic underlying intuition is illustrated by issues in biomedical imaging[45, 78]. Formally, we would like to make a connection between Morse theory and Massey products. Let us begin by reviewing the fundamentals of Morse theory[61, 14, 34, 62].

Consider a compact n -dimensional manifold M , denote the i^{th} Betti number (i.e. the rank of the i^{th} homology group) by $\beta_i(M)$, and form the Poincaré polynomial:

$$P_t(M) \triangleq \sum_{k=0}^n t^k \beta_k(M) \quad (8)$$

The critical points, $\{p_i\}$, of a function f , mapping M to \mathbb{R}^1 , are said to be nondegenerate if the Hessian matrix (i.e. matrix of second partial derivatives) has full rank at every critical point. A function with nondegenerate critical points is called a Morse function. (Morse functions are not rare in that they form an open dense set in the space of all functions.)

Given a Morse function on M , we define the index of the j^{th} critical point, p_j , to be the number of negative eigenvalues of the Hessian matrix evaluated at p_j . (Note: the eigenvalues are real because the Hessian is symmetric, and nonzero since p_j is nondegenerate.) Let m_i be the number of critical points of index i , and form the Morse polynomial:

$$\mathcal{M}_t(f) = \sum_{k=0}^n t^k m_k(f) \quad (9)$$

A fundamental result of Morse theory asserts that

$$\mathcal{M}_t(f) - P_t(M) = (1+t) Q_t(f) \quad (10)$$

where $Q(f)$ is a polynomial with nonnegative integer coefficients. An equivalent formulation (the Morse inequalities) is found by equating powers of t , solving for the coefficients of $Q_t(f)$ and using the nonnegativity of these coefficients to obtain the inequalities:

$$\begin{aligned} m_0 &\geq \beta_0 \\ m_1 - m_0 &\geq \beta_1 - \beta_0 \\ m_2 - m_1 + m_0 &\geq \beta_2 - \beta_1 + \beta_0 \\ &\vdots \geq \vdots \end{aligned} \quad (11)$$

Given the Morse Inequalities expressed in the form of equation (10), two strategies for their use emerges:

1. If $P_t(M)$ is known, then one has a lower bound of the number of critical points of every index. Specifically, a very loose bound is given by $m_i \geq \beta_i$ for all i .
2. If no consecutive powers of t occur in the Morse polynomial then $Q_t(f) = 0$ and so $\mathcal{M}_t(f) = P_t(M)$. In this case knowledge of the critical points gives complete knowledge of the Betti numbers.

A function for which $\mathcal{M}_t(f)$ is equal to $P_t(M)$ is called a perfect Morse function. In general there are topological obstructions to finding a perfect Morse function. For example, the Morse inequalities are also valid if homology groups are computed using coefficients from any field (e.g. \mathbb{Z}_p where p is a prime). Hence a perfect Morse function is only possible when the homology groups of M are torsion free.

With this background we can return to the topic of Massey products. Concretely I would like to conjecture that the obstruction to finding a perfect Morse function depends on Massey products (just as the obstruction to globally defined Clebsch potentials seems to). Specifically, I have a conjecture:

$$Q_t(f) = 0 \stackrel{?}{\Rightarrow} \left\{ \begin{array}{l} \text{All Massey products in the} \\ \text{cohomology ring of } M \text{ are trivial} \end{array} \right\} \quad (12)$$

I have bits of circumstantial evidence to support this conjecture. I will present them before I proceed to outline how I may go about proving (12).

1. The construction of perfect Morse functions plays a great role in computing Poincaré polynomials in various contexts. For example Yang-Mills Theory on Riemann Surfaces, stable homotopy groups of Lie groups[14], Grassmannians, etc. These examples seem to indicate that the above conjecture is overly ambitious. However, in these cases the underlying manifold has a Kähler structure, and it is known that Kähler manifolds have vanishing Massey Products[39]. Hence, in order to make progress with the above conjecture, we can search the literature for answers to the following two questions:

- (a) Do all Kähler manifolds admit a perfect Morse function? If not what are the obstructions?
- (b) Are there classes of non-Kähler manifolds which are known to admit perfect Morse functions?

2. In both the calculus of variations and finite dimensional critical point theory, the "mountain pass lemma" is frequently used to exhibit saddle point extrema. (This is a powerful technique in the theory of nonlinear elliptic equations[81]). Underlying the proof of the mountain pass lemma is a linking argument which leads to deeper results. The notion of linking describes the simplest of cohomology operations and Massey products appear to be the right framework for generalized results in critical point theory.

To see how Massey products can come into Morse Theory naturally, we have to recall how the polynomial $Q_i(f)$ arises in the first place. The level sets of a function $f: M \rightarrow \mathbb{R}$ define a "stratification" of the manifold M . Let us perturb f such that all critical points are nondegenerate and each level set contains at most one critical point. Furthermore, define submanifolds

$$X_c = \{x \in M \mid f(x) \leq c\} \quad (13)$$

If B^λ denotes the λ -dimensional unit ball whose boundary is the $(\lambda - 1)$ -dimensional sphere $S^{\lambda-1}$, and \sim is used to denote a homotopy equivalence, then it is a basic result[14, 34, 61], that for sufficiently small $\varepsilon > 0$, we have the following "change in topology" formula:

$$X_{c+\varepsilon} \sim \begin{cases} X_{c-\varepsilon} & \text{if } c \text{ is not a critical value} \\ X_{c-\varepsilon} \cup B^\lambda & \text{if } c \text{ is a critical value tied to a critical point } p \\ & \text{of index } \lambda. \end{cases} \quad (14)$$

To see how the polynomial $Q_i(f)$ emerges we now appeal to algebraic topology. Consider the above "change in topology formula" for each critical point p_j and corresponding critical value c_j . Introduce the inclusion and projection maps:

$$\left. \begin{array}{l} i_j : X_{c_j-\varepsilon} \longrightarrow X_{c_j+\varepsilon} \\ \pi_j : X_{c_j+\varepsilon} \longrightarrow X_{c_j+\varepsilon}/X_{c_j-\varepsilon} \end{array} \right\} \quad (15)$$

and consider the long exact sequence of homology groups arising at the j^{th} critical point:

$$\begin{array}{c}
\cdots \\
\curvearrowright \\
H_{k-1}(X_{c_j-\epsilon}) \xrightarrow{i_{j,k-1}} H_{k-1}(X_{c_j+\epsilon}) \xrightarrow{\pi_{j,k-1}} H_{k-1}(X_{c_j+\epsilon}, X_{c_j-\epsilon}) \quad \delta_{j,k-1} \\
\curvearrowright \\
H_k(X_{c_j-\epsilon}) \xrightarrow{i_{j,k}} H_k(X_{c_j+\epsilon}) \xrightarrow{\pi_{j,k}} H_k(X_{c_j+\epsilon}, X_{c_j-\epsilon}) \quad \delta_{j,k} \\
\curvearrowright \\
\cdots
\end{array} \tag{16}$$

where $i_{j,l}$ and $\pi_{j,l}$ are induced mappings on the l th homology groups and $\delta_{j,l}$ is the "connecting homomorphism." If we define the Poincaré polynomial of the connecting homomorphism as

$$P(\text{Im } \delta_j) = \sum_{k=0}^n \dim(\text{Im } \delta_{j,k+1}) t^k \tag{17}$$

then, by standard techniques[34], we have

$$P(X_{c_j+\epsilon}, X_{c_j-\epsilon}) = P(X_{c_j+\epsilon}) - P(X_{c_j-\epsilon}) + (1+t) P(\text{Im } \delta_j) \tag{18}$$

We will now outline how the Morse inequalities follow if we extract the algebraic content of equation (14) and consider equation (18) for each critical point and sum. First, from equation (14) we deduce a homotopy equivalence

$$X_{c_{(j-1)+\epsilon}} \sim X_{c_j-\epsilon} \tag{19}$$

and so, by the homotopy invariance of homology groups, we have

$$\left. \begin{array}{l} H_k(X_{c_{(j-1)+\epsilon}}) \simeq H_k(X_{c_j-\epsilon}) \text{ for all } k \\ \text{and } P_t(X_{c_{(j-1)+\epsilon}}) = P_t(X_{c_j-\epsilon}) \end{array} \right\} \tag{20}$$

Notice that if f has N critical points $X_{c_0-\epsilon}$ is the empty set and $X_{c_N+\epsilon}$ is M . Equation (20) then shows that if we consider a copy of equation (18) for each critical point and sum, the first two terms on the right hand side form a telescoping sum so that

$$\sum_{j=1}^N P_t(X_{c_j+\epsilon}, X_{c_j-\epsilon}) = P_t(M) + (1+t) \sum_{j=1}^N P_t(\text{Im } \delta_j) \tag{21}$$

Next, we continue to extract algebraic consequences of equation (14). As we cross a critical value, B^{λ_j} in equation (14) is attached to $X_{c_j-\epsilon}$ along an S^{λ_j-1} and this provides us with a relative homotopy equivalence which descends to an isomorphism of homology groups and equality of Poincaré polynomials:

$$\left. \begin{array}{l} (X_{c_j+\epsilon}, X_{c_j-\epsilon}) \sim (S^{\lambda_j}, \text{pt.}) \text{ implies} \\ H_k(X_{c_j+\epsilon}, X_{c_j-\epsilon}) \simeq H_k(S^{\lambda_j}, \text{pt.}) \text{ for all } k, \\ \text{and so } P_t(X_{c_j+\epsilon}, X_{c_j-\epsilon}) = P_t(S^{\lambda_j}, \text{pt.}) = t^{\lambda_j} \end{array} \right\} \tag{22}$$

Substituting this expression for the Poincaré polynomial into the left hand side of equation (21) and collecting terms with same index, the left hand side is seen to be the Morse polynomial defined by equation (9). Hence

$$\mathcal{M}_t(f) = P(M) + (1+t) \sum_{j=1}^N P(\text{Im } \delta_j) \quad (23)$$

At this point we recover the Morse inequalities (10) and $Q_t(f)$ can be identified with the summation on the right hand side - but we can do better!

Given the homology groups of the sphere modulo a point, equation (22) shows that $H_k(X_{c_j+\epsilon}, X_{c_j-\epsilon})$ is trivial unless $k=\lambda_j$. Hence, in the exact sequence (16), the connecting homomorphisms are all trivial except for possibly δ_{j,λ_j} . This means equation (17) can be simplified to:

$$P_t(\text{Im } \delta_j) = \dim(\text{Im } \delta_{j,\lambda_j}) t^{\lambda_j-1} \quad (24)$$

Substituting this into (23) and identifying $Q_t(f)$ gives

$$Q_t(f) = \sum_{j=1}^N \dim(\text{Im } \delta_{j,\lambda_j}) t^{\lambda_j-1} \quad (25)$$

At this point we can return to the conjecture (12). The conjecture amounts to saying that if we compute the cohomology of the manifold M by using a cell decomposition derived from the critical points and level sets of a Morse function, then nontrivial Massey products in the cohomology ring should force $\dim(\text{Im } \delta_{j,\lambda_j})$ to be nonzero for some collection of j 's. To make this precise, we propose to compute the cohomology of M by appealing to a spectral sequence arising from the cell decomposition of M and to reexpress $Q_t(f)$ in terms of the differentials of the spectral sequence. It is known that Massey products are naturally related to the differentials of a spectral sequence[59] and, hopefully we can obtain deeper insight into conjecture (12) by pursuing the spectral sequence framework.

Let us consider two other contexts which shed light on conjecture (12):

3. Further insight into conjecture (12) can be obtained by considering the obstruction to finding a perfect Morse function on a compact, connected, orientable, three dimensional manifold M^3 . To start, consider two embeddings, M_1 and M_2 , into \mathbb{R}^3 , of a compact, connected, two-dimensional manifold M_g^2 of genus g . The embeddings force M_i ($i = 1, 2$) to be orientable and hence they have an "inside" and an "outside." Call the insides H_1 and H_2 so that

$$\partial H_i = M_i \quad (i = 1, 2) \quad (26)$$

The H_i 's are called handle bodies. If we have a diffeomorphism $\alpha : M_1 \rightarrow M_2$ then we can glue H_1 to H_2 along their boundaries to get a three manifold M^3 :

$$M^3 = H_1 \underset{\alpha}{\cup} H_2 \quad (27)$$

It is a basic result in three manifold theory that any connected, compact, orientable 3-Manifold M^3 admits such a splitting for some g [76]. This is called a Heegaard splitting of M^3 . The minimal genus g for which a given manifold M^3 has a Heegaard splitting is a topological invariant called the genus of M^3 and is denoted here by $g^*(M^3)$. The genus of M^3 has a definition in terms of the fundamental group of M^3 .

We next consider Heegaard splittings which arise from Morse functions. Consider first the following

Basic Fact: Th^m11.3[34]

Let M^n be a smooth compact connected n -dimensional manifold. Then, on M^n , there is an admissible Morse function f , where $0 \leq f \leq n$, such that it has only one minimum (point of index 0), only one maximum (point of index n), and each critical point of index λ is situated on the surface $f^{-1}(\lambda)$.

Note that this statement implicitly defines the notion of an admissible Morse function. As a corollary, we see that an admissible Morse function defines a Heegaard splitting. Concretely, the Morse polynomial of an admissible Morse function $f : M^3 \rightarrow \mathbb{R}$ looks like:

$$\mathcal{M}_t(f) = 1 + (1+t)t g(f) + t^2 \quad (28)$$

where $g(f)$ is the genus of the two dimensional manifold $f^{-1}(\frac{3}{2})$ and the "handle bodies" of the Heegaard splitting are given by:

$$\begin{aligned} H_1 &= \{x \in M \mid f(x) \leq \frac{3}{2}\} \\ H_2 &= \{x \in M \mid f(x) \geq \frac{3}{2}\}. \end{aligned} \quad (29)$$

In this case, the Poincaré Polynomial looks like

$$P_t(M^3) = 1 + (1+t)t \beta_1(M^3) + t^3 \quad (30)$$

Equations (10), (28), and (30) enable us to compute $Q_t(f)$:

$$\begin{aligned} (1+t) Q_t(f) &= \mathcal{M}_t(f) - P_t(M^3) = (1+t)t (g(f) - \beta_1(M^3)) \\ \text{i.e. } Q_t(f) &= t (g(f) - \beta_1(M^3)) \end{aligned} \quad (31)$$

Equations (25) and (31) give us the nontrivial result:

$$g(f) - \beta_1(M^3) = \sum_{j \ni \lambda_j=2} \dim(\text{Im } \delta_{j,\lambda_j}) \quad (32)$$

By definition, for any admissible Morse function we have

$$g(f) - \beta_1(M^3) \geq g^*(M^3) - \beta_1(M) \geq 0 \quad (33)$$

The fact that $g^*(M^3) > \beta_1(M^3)$ for most 3-manifolds shows us that perfect admissible Morse functions can rarely be expected to exist. Hence, to shed light on conjecture (12) we can consider three questions:

- Is there any obstruction to finding an admissible Morse function for which

$$g(f) = g^*(M^3)?$$

- Is $g^*(M^3) - \beta_1(M^3)$ expressible in terms of the cohomology ring of M^3 and in particular, the “number” of nonzero Massey products? (Valuable insight might be obtained by relating $g^*(M^3)$ to $\pi_1(M^3)$ to Milnor’s μ -invariants, to Massey products)
 - How can alternate definitions of $g^*(M^3)$ along with equations (32) and (33) be exploited?
4. Finally, in order to relate Massey products to Morse theory, we may try to relate Witten’s proof of the Morse inequalities[92], the heat equation method for harmonic forms[30], in the context of Witten’s deformed Laplacian, and John Lott’s work on the asymptotics of the heat kernel and its relation to Massey products[55].

This connection may also have implications for molecular biology[91, 85, 32].

3 Focus of ongoing research

3.1 Resolvable intermediate problems regarding Clebsch potentials

Surveying the literature, it is clear that there is no “usable answer” to the question of finding the topological obstruction to having globally defined Clebsch potentials. Furthermore, the engineering and physics literature is riddled with papers which are completely erroneous when it comes to articulating the global existence of Clebsch potentials. On the other hand, the mathematics literature, in particular the theory of G-structures[25], contact structures[8], and foliations contains the formalism to clearly articulate the problem, and possibly find a solution (even if it is crouched in algebraic structures which are not easily reduced to numerical computation). For this reason, I have formulated several intermediate problems which bridge the gap between the classical local results and the problems addressed by the modern formalism.

Given a “nice” region R in \mathbb{R}^3 and a divergence zero vector field G in R , assume that we can write

$$G = \text{curl } F \text{ in } R \quad (34)$$

The obstruction to doing so depends on the cohomology group $H^2(R; \mathbb{R})$ which we assume to be trivial. Next, for simplicity, suppose

$$F \cdot \text{curl } F \neq 0 \text{ in } R \quad (35)$$

We can then describe R by an atlas \mathcal{A} specified by a Monge potential representation. That is, the atlas \mathcal{A} is a collection of charts:

$$(U_i, \tilde{r}_i(\xi_i, \eta_i, \phi_i)) \quad 1 \leq i \leq n(\mathcal{A}) \quad (36)$$

where, $n(\mathcal{A})$ is the number of charts in the atlas and, by the definition of an atlas, the bijections \tilde{r}_i satisfy

$$(\tilde{r}_i)^{-1} : U_i \rightarrow \mathbb{R}^3 \quad (37)$$

Furthermore, by the definition of Monge potentials, we have

$$F = \xi_i \text{grad } \eta_i + \text{grad } \phi_i \text{ on } U_i \quad (38)$$

Taking the curl we have

$$G = \text{curl } F = \text{grad } \xi_i \times \text{grad } \eta_i \text{ on } U_i \quad (39)$$

and, by (35) we have:

$$F \cdot \text{curl } F = \frac{\partial(\xi_i, \eta_i, \phi_i)}{\partial(x, y, z)} \neq 0 \text{ on } U_i \quad (40)$$

which coincides with what we expect from the definition of a chart. At this point, it is very fruitful to pause and consider the geometric consequences of what we have done. Equation (40) implies

$$\frac{\partial(\xi_i, \eta_i, \phi_i)}{\partial(\xi_j, \eta_j, \phi_j)} = 1 \text{ on } U_i \cap U_j \quad (41)$$

while equation (6) implies

$$\frac{\partial(\xi_i, \eta_i)}{\partial(\xi_j, \eta_j)} = 1 \text{ on } U_i \cap U_j \quad (42)$$

From this vantage we see that there are several interesting and relatively simple questions which we may ask.

Question 1 Given (R, \bar{F}) as above, what is the minimum, over all possible atlases, of the number of charts in an atlas.

Clearly the above number gives a measure of "how impossible" it is to find a set of global Clebsch potentials for F on R .

Equations (41) and (42) above tie our problem to the theory of G-structures, and the integer which answers question 1 is the analogue of the "Liusternik-Snirelman Category" for

some cohomology theory cooked up from the constraints on the differential structure on our atlas.

A natural variant of the above problem is the following. Consider the minimum of the answer to question 1 over all divergence free vector fields F which satisfy equation (35). If R is a closed three-dimensional manifold then this yields an interesting topological invariant. This invariant is intimately related to the work of Arnold[2], Moffatt[70, 71] and Freedman[36] on using helicity, (mean asymptotic linking number) and mean asymptotic crossing numbers to get lower bounds on the energy of ideal magnetohydrodynamic flows.

Another intermediate question is:

Question 2 Can we use equation (42) to try to build a differential structure for a surface? (Obstructions would also be obstructions to having a set of globally defined Clebsch potentials).
If so, can we then use the " ϕ coordinate" to build a three dimensional manifold? That is, can we build a foliation out of surfaces " $\phi = \text{const?}$ "

This question ties back to the use of Helicity to define an obstruction to globally defined Clebsch potentials.

My previous work in micromagnetics[50, 51], coupled with the work of Arnold[2, 4, 23, 24, 46, 86], Freedman[35, 36, 37], Moffatt[65] and others, indicate that a fruitful question to consider is the time evolution of a set of Clebsch potentials. Specifically, in the "accessibility problem", where a vector field F is given throughout R at two time instants t_1 and t_2 and on ∂R for $t \in [t_1, t_2]$, one asks if there exists a set of Clebsch potentials on $R \times [t_1, t_2]$. The starting point of such an investigation is to derive a formula which says

$$\int_R \bar{F} \cdot \text{curl } \bar{F} dV \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\partial R} (\text{something}) dS dt = \begin{array}{l} \text{number of} \\ \text{"flux lines reconnected"} \\ \text{in } R \times [t_1, t_2] \end{array}$$

In other words, let $I = [t_1, t_2]$, construct some 3-form ω out of the given data on $\partial(R \times I)$, and, without the hypotheses given by equation (35), the above formula says:

$$\int_{\partial(R \times I)} \omega = \sum \text{oriented flux line reconnections in } R \times I$$

3.2 Algebraic gadgets which reveal 3-D complexity

The obstruction to finding a set of global Clebsch potentials is ultimately related to the twisting and linking of flux lines, and the task at hand is simplified if we can find algebraic structures which articulate this behavior. So far we have considered linking numbers and Massey products in cohomology. I have found that a useful question to ask is: "How uncontractible" is a given region of space? It turns out that Massey products, K.T. Chen's iterated integrals[21], loop space homology, Dennis Sullivan's minimal models[84], the lower central series of link groups, and Thurston's norm on homology[87] all come to play and are

intimately related in this context. To point out these relationships, we consider the Poincaré homomorphism

$$P : \pi_1(R) \rightarrow H_1(R; \mathbb{Z}) \quad (43)$$

which sends the homotopy class of a loop in R to its homology class. The map is onto and its kernel is the commutator subgroup $[\pi_1, \pi_1]$. Hence we have the isomorphism

$$\pi_1 / [\pi_1, \pi_1] \simeq H_1 \quad (44)$$

This isomorphism plays a key role in understanding counterintuitive aspects of making "cuts" for magnetic scalar potentials in three dimensions[40]. Our immediate concern, is to find commutative algebraic gadgets, more complicated than H_1 , which help articulate the complexity of π_1 . (In general, for π_1 's coming from three dimensional manifolds, it is not known whether the basic group theoretic questions are "decidable" in the computer science sense of the word.) We begin by considering the graded group $\text{gr}(\pi_1)$ constructed out of π_1 , and as a preliminary step, consider the lower central series of a group G , defined recursively by

$$G_1 = G, \quad G_{n+1} = (G_n, G), \quad n = 1, 2, \dots \quad (45)$$

where (\cdot, \cdot) denotes a commutator. The associated graded group is formally defined as

$$\text{gr}(G) = \bigoplus_{r=1}^{\infty} G^r / G^{r+1} \quad (46)$$

and for a given t the object (see[41])

$$\text{gr}(G) / G^{t+1} = \bigoplus_{r=1}^t G^r / G^{r+1} \quad (47)$$

is of great interest. It is a nilpotent \mathbb{Z} -graded Lie algebra. In particular, given $R \in \mathbb{R}^3$, we can use equation (44) and the following exact sequence[31, 83]

$$H_2(R; \mathbb{Z}) \xrightarrow{\mu} \wedge^2 H_1(R; \mathbb{Z}) \xrightarrow{[\cdot, \cdot]} ([\pi_1, \pi_1] / [\pi_1, [\pi_1, \pi_1]]) \rightarrow 0 \quad (48)$$

where:

$$\begin{aligned} \pi_1 &= \pi_1(R) \\ [\cdot, \cdot] &\text{ is induced by the commutator and} \\ \mu &\text{ is dual to } \cup: \wedge^2 H_1(R; \mathbb{Z}) \rightarrow H_2(R; \mathbb{Z}) \end{aligned}$$

to obtain

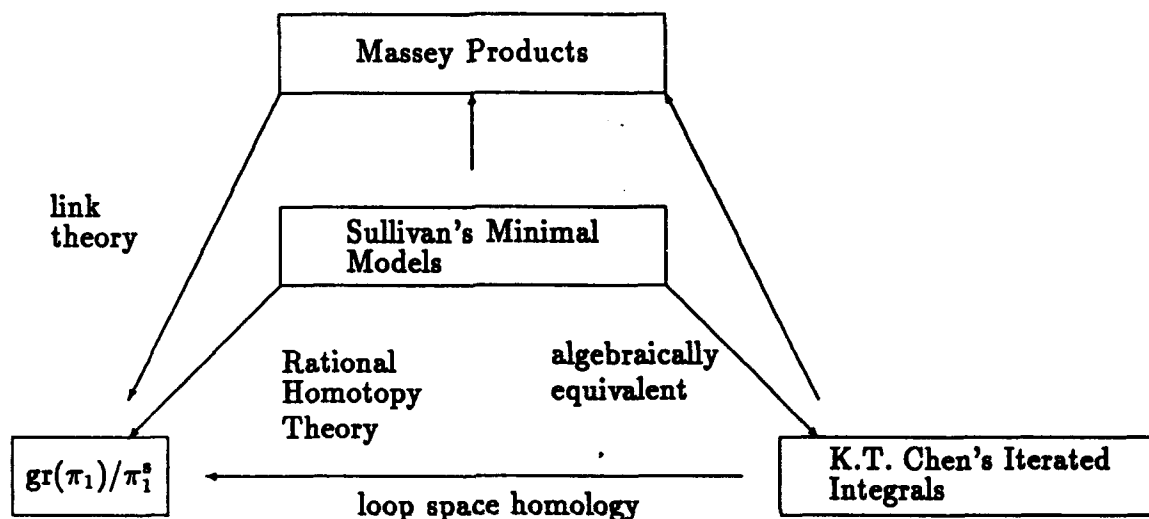
$$\text{gr}(\pi_1) / \pi_1^{(2)} = \pi_1 / [\pi_1, \pi_1] = H_1 \quad (49)$$

$$\text{gr}(\pi_1) / \pi_1^{(3)} = H_1 \oplus \text{Image}([\cdot, \cdot]) \quad (50)$$

These relations have already been considered in my work on eddy currents in multiply connected regions. In order to consider obstructions to Clebsch potentials, we want to consider

$$\text{gr}(\pi_1(\tilde{R})) / \pi_1(\tilde{R})^{(l+1)} \quad (51)$$

where $\tilde{R} \subset R$ is the exterior of a flux tube or some other region left invariant by the "flow" of curl F . The literature cited in the bibliography[11, 15, 16, 17, 18, 19, 20, 21, 22, 28, 29, 33, 38, 39, 41, 42, 43, 58, 75, 77, 79, 80, 82, 83, 84] makes the following connections:



The additional algebraic gadget which appears in three dimensional problems is Thurston's norm on homology[87]. Its relation to the above diagram is less obvious and does not seem to appear in the literature.

3.3 Clebsch potentials and Hamiltonian systems

It was pointed out to me by Professor Marsden [U.C. Berkeley] that a solenoidal vector field F possessing a set of Clebsch potentials ξ and η implies a Hamiltonian structure for

$$\dot{x} = F(x) \quad (52)$$

To see this, consider $\phi(x)$ and compute

$$\frac{d\phi}{dt} = \nabla\phi \cdot \frac{dx}{dt} = \nabla\phi \cdot (\nabla\xi \times \nabla\eta) \quad \text{by (52)} \quad (53)$$

$$\text{or} \quad \dot{\phi} = \nabla\eta \cdot (\nabla\phi \times \nabla\xi)$$

Alternatively, we can write

$$\dot{\phi} = \{\phi, \xi\}_\eta \quad (54)$$

where we define

$$\{a, b\}_\eta = \nabla \eta \cdot (\nabla a \times \nabla b) \quad (55)$$

and it is easily verified that the bracket $\{ \quad \}$ defines a Poisson structure. This brings us back to the theory of contact structures. That is the global theory of Hamiltonian systems can be used to define obstructions to finding Clebsch potentials.

From this vantage, it is clear that one must retreat from the overly ambitious goal. For the antenna synthesis problem, it is reasonable to look for additional constraints on the current distribution, which make it more amenable to description by a global set of Clebsch potentials. Unfortunately, no such constraints are known. Alternatively, one could look to other models of physical phenomena involving solenoidal vector fields (e.g. micromagnetics or inviscid, incompressible fluid flow)[1, 60] to find simpler Hamiltonian systems which can guide our intuition.

4 Appendices

4.1 Paper published in which AFOSR support is acknowledged

"Metric Dependent Aspects of Inverse Problems and Functionals Based on Helicity"

4.2 Previous work related to topological accessibility

"A Topological Invariant for the Accessibility Problem of Micromagnetics"

Metric dependent aspects of inverse problems and functionals based on helicity

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The helicity of a vector field is a metric independent density. Functionals with first order elliptic systems for Euler-Lagrange equations have been constructed from the helicity. The metric invariance is preserved for finite element discretizations involving "Whitney elements." This paper relates differential geometric aspects of inverse problems to helicity based functionals in two contexts. First, the inverse problem of electrical impedance tomography in isotropic media is known to be equivalent to determining a metric within a given conformal class from a given "Dirichlet to Neumann" map. This fact is related to the helicity functional and Wexler's algorithm for recovering an isotropic conductivity. Second, Maxwell's equations in "spinor form" are shown to be the Euler-Lagrange equations of some complexified time dependent generalization of the helicity functional. In this case metric dependent aspects yield insight into the "inverse kinematic problem in seismology." These two examples illustrate the underlying geometric structure in classes of inverse problems and algorithms for their solution.

"Do you know Grassmann's Ausdehnungslehre? Spottiswood spoke of it in Dublin as something above and beyond quaternions. I have not seen it, but Sir William Hamilton of Edinburgh used to say that the greater the extension the smaller the intention."

"May one plough with an ox and an ass together? The like of you may write everything and prove everything in 4nions, but in the transition period the bilingual method may help to introduce and explain the more perfect."

Excerpt of a letter from J. C. Maxwell to P. G. Tait; A. P. Wills, *Vector Analysis with an Introduction to Tensor Analysis* (Prentice-Hall, Englewood Cliffs, NJ, 1931), pp. XXV-XXVI.

I. INTRODUCTION

The issues to be addressed are most easily introduced by recalling the metric and coordinate invariance of the helicity of the magnetic field intensity vector \mathbf{H} . Since Ampère's law associates the magnetic field intensity \mathbf{H} with integration along curves, we naturally identify it with a differential 1 form as follows:

$$\omega = \mathbf{H} \cdot d\mathbf{r}. \quad (1)$$

The helicity is then defined to be

$$I(\omega) = \int_R \omega \wedge d\omega = \int_R \mathbf{H} \cdot \text{curl } \mathbf{H} \, dV, \quad (2)$$

where R is a three-dimensional region. The differential form point of view¹ is inflicted on the reader at the onset since the definition (2) is clearly metric and coordinate invariant.² Furthermore, these properties can be preserved in the context of the finite element method by appealing to Whitney forms.³ To see this, consider a tetrahedron Δ with vertices denoted by p_i , $1 \leq i \leq 4$, and define edge variables h_{ij} by

$$h_{ij} = \int_{p_i}^{p_j} \omega. \quad (3)$$

Since $h_{ij} = -h_{ji}$, we can consider the h_{ij} 's to be the components of a skew-symmetric 4×4 matrix \mathcal{H} . Given these edge variables, we can find a piecewise linear approximation $\tilde{\omega}$ to ω by appealing to Whitney forms:

$$\tilde{\omega} = \sum_{i,j=1}^4 h_{ij} \xi_i d\xi_j. \quad (4)$$

Here ξ_i , $1 \leq i \leq 4$, are the usual barycentric coordinates on Δ . When discretized, the contribution to the helicity from one tetrahedron takes on the amazingly simple form⁴:

$$3 \int_{\Delta} \tilde{\omega} \wedge d\tilde{\omega} = \text{Pfaff}(\mathcal{H}), \quad (5)$$

where, given an orientation for Δ , $\text{Pfaff}(\mathcal{H})$ is a square root of $\det(\mathcal{H})$:

$$\text{Pfaff}(\mathcal{H}) = h_{12}h_{34} - h_{13}h_{24} + h_{14}h_{23}. \quad (6)$$

Equations (5) and (6) are independent of interpolation node coordinates and any metric dependent bilinear pairing such as a constitutive law. Furthermore it is clear that the quadratic form associated with the "element stiffness matrix" can be evaluated with only three multiplications!

The helicity functional²

$$J(\mathbf{H}, \lambda) = \int_R \mathbf{H} \cdot (\frac{1}{2} \text{curl } \mathbf{H} + \mu \text{grad } \lambda - \mathbf{J}) dV \quad (7)$$

exploits the metric invariance of helicity. Identifying the current density J with a differential 2 form, the helicity functional is seen to be coordinate-independent except for the term involving the Lagrange multiplier λ which ensures that μH is a solenoidal vector field. Again this property is preserved when the functional is discretized using Whitney forms and the discrete Euler-Lagrange equations have a remarkably simple interpretation in terms of Ampere's Law.⁵ This minimal dependence on constitutive laws and coordinates makes the Helicity functional ideal for problems requiring iterative solution where either the mesh is deformed or the constitutive law is updated at every iteration. This aspect will be developed in Sec. II when we consider Wexler's Algorithm for Impedance Computed Tomography.

To introduce a complexified time dependent helicity functional, consider first the Euler-Lagrange equation for the functional (7):

$$D \begin{pmatrix} H \\ \lambda \end{pmatrix} = \begin{pmatrix} \text{curl } H + \mu \text{ grad } \lambda \\ -\text{div}(\mu H) \end{pmatrix} = \begin{pmatrix} J \\ 0 \end{pmatrix}. \quad (8)$$

If we choose Cartesian coordinates and set $\mu = 1$, D satisfies

$$D^2 = -I\nabla^2, \quad (9)$$

where I is the 4×4 identity matrix.² Thus D is an elliptic partial differential operator which, in some concrete sense, is the square root of the positive definite Laplace operator. The spectrum of D is the square root of that of the Laplace operator and square roots are distributed in a manner which is nearly symmetric about zero. This has interesting consequences for the numerical solution of the discretized equations.⁶ If we define an operator \bar{D} by

$$\bar{D} = D + iI \frac{\partial}{\partial t}, \quad (10)$$

then using Eq. (9), it is easy to verify that

$$\bar{D}\bar{D}^* = I \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right). \quad (11)$$

Thus, \bar{D} is a partial differential operator which, in some concrete sense is the square root of the wave operator. Note that \bar{D} now needs to operate on pairs consisting of a complex valued vector and a complex valued scalar. Although the operator \bar{D} is closely related to the Dirac operator and hence the quantum mechanical view of magnetism, our purpose is to extend previous work⁷ and consider how a related operator reproduces Maxwell's equations while elucidating the coordinate dependence of the associated complexified time dependent helicity functional whose Euler-Lagrange operator closely resembles \bar{D} . Thus, Sec. III of this paper considers a complexified time dependent helicity functional whose Euler-Lagrange equations are related to Maxwell's equations just as Eqs. (8) are related to magnetostatics. This functional is then considered in the context of inverse problems.

II. WEXLER'S ALGORITHM REVISITED

Consider a region R containing a given inhomogeneous isotropic conductivity σ . Steady current flow is governed by

$$\text{div } J = 0, \quad J = \sigma E, \quad \text{curl } E = 0 \quad \text{in } R. \quad (12)$$

Letting E be minus the gradient of the scalar potential ϕ , we can prescribe a boundary potential ϕ_B on ∂R and rephrase the determination of J in terms of a Dirichlet problem for ϕ :

$$-\text{div}(\sigma \text{ grad } \phi) = 0 \quad \text{in } R, \quad (13)$$

$$\phi = \phi_B \quad \text{on } \partial R. \quad (14)$$

Given a solution to the above boundary value problem, we can calculate the Neumann data $\sigma \hat{n} \cdot \text{grad } \phi$ and recover the normal component of J on ∂R . Alternatively, instead of the Dirichlet data (14), J can be uniquely determined by imposing the Neumann data on ∂R . From a variational point of view, the inhomogeneous Neumann data is difficult to impose and, when R is two-dimensional, the Neumann problem for ϕ is often reformulated as a Dirichlet problem for a stream function. Similarly, in three dimensions, we can express J as the curl of a vector potential T . Up to a "gauge transformation" T is just the magnetic field intensity H and the situation is analogous to the use of a vector potential in magnetostatics. By imposing the tangential components of T on ∂R , the normal component of J is implicitly prescribed. When J and E are eliminated in Eq. (12) through the use of T , current boundary conditions are imposed, and the "gauge" of T is fixed, we end up with

$$\text{curl} \left(\frac{1}{\sigma} \text{curl } T \right) = 0 \quad \text{in } R, \quad (15)$$

$$\text{div}(\sigma T) = 0 \quad \text{in } R, \quad (16)$$

$$\hat{n} \times T_1 = \hat{n} \times T_B \quad \text{on } \partial R. \quad (17)$$

In order to avoid confusion, we call Eqs. (14) and (17), voltage and current boundary conditions, respectively.

For a given inhomogeneous conductivity σ , there is a unique correspondence:

$$\phi \text{ on } \partial R \text{ mod constants} \leftrightarrow \hat{n} \cdot \text{curl } T \text{ on } \partial R \quad (18)$$

which we call the voltage to current map. Impedance tomography seeks to recover σ given this voltage to current map. In practice, given a collection of voltage and current measurements $\{\phi_{B_i}, \hat{n} \cdot \text{curl } T_{B_i}\}_{i=1}^N$, we would like to recover a reasonable estimate for σ in R . Reference 8 seems to have proposed the most useful algorithm for doing this. See Kohn and Vogelius⁹ for a survey of early results, and Sylvester and Uhlmann¹⁰ for a geometric understanding of the problem. Our concern is to understand how the helicity functional points to a useful variant of Wexler's algorithm.

Consider the helicity functional:

$$I(T, \phi) = \int_R T \cdot \left(\frac{1}{2} \text{curl } T + \sigma \text{ grad } \phi \right) dV. \quad (19)$$

The Euler-Lagrange equations are

$$\text{curl } \mathbf{T} + \sigma \text{ grad } \phi = 0, \quad (20)$$

$$-\text{div}(\sigma \mathbf{T}) = 0. \quad (21)$$

Taking the divergence of Eq. (20) yields Eq. (13) while dividing Eq. (20) by σ and taking the curl yields Eq. (15). Equation (21) is the same as Eq. (16). Hence any pair (\mathbf{T}, ϕ) satisfying the Euler-Lagrange equations of the helicity functional (19) automatically satisfy the partial differential equations (but not boundary conditions!) of steady current conduction when formulated in terms of vector or scalar potentials. Furthermore, recalling Eq. (9) we see in what sense Eqs. (20) and (21) are the square root of the combination of both potential formulations.

We now demonstrate why the metric invariance of helicity is so useful for three-dimensional impedance tomography. Suppose we have the data $\{\phi_n, \hat{n} \cdot \text{curl } \mathbf{T}_n\}_{n=1}^N$ of this inverse problem and suppose that from this data we find a consistent set of pairs of boundary conditions for the helicity functional $\{(\phi_n, \hat{n} \cdot \mathbf{T}_n), (\hat{n} \times \mathbf{T}_n)\}_{n=1}^N$. Furthermore, let the functional (19) with principal boundary conditions $(\phi_n, \hat{n} \cdot \mathbf{T}_n)$ and $(\hat{n} \times \mathbf{T}_n)$ be denoted by I_{σ, \mathbf{T}_n} and I_{σ, \mathbf{I}_n} , respectively. Given an initial "guess" σ_0 for the conductivity, we can find the stationary points of the functionals $I_{\sigma_0, \mathbf{T}_n}$ and $I_{\sigma_0, \mathbf{I}_n}$, where $1 \leq n \leq N$. In general, after k iterations of Wexler's updating scheme, we can let $\{\mathbf{T}_{k, \mathbf{T}_n}, \phi_{k, \mathbf{T}_n}\}$ and $\{\mathbf{T}_{k, \mathbf{I}_n}, \phi_{k, \mathbf{I}_n}\}$ be the stationary points of $I_{\sigma_k, \mathbf{T}_n}$, respectively. The rule for updating σ is then given by

$$\sigma_{k+1} = \frac{\sum_{n=1}^N \text{curl } \mathbf{T}_{k, \mathbf{I}_n} \cdot \text{grad } \phi_{k, \mathbf{T}_n}}{\sum_{n=1}^N \text{grad } \phi_{k, \mathbf{T}_n} \cdot \text{grad } \phi_{k, \mathbf{T}_n}}. \quad (22)$$

Hence every iteration of the proposed algorithm involves finding the stationary values of $2N$ functionals $I_{\sigma_k, \mathbf{T}_n}$, $I_{\sigma_k, \mathbf{I}_n}$, $1 \leq n \leq N$ and updating σ by the rule (22). The surprising aspect is that the $2N$ functional all depends on σ_k in the same way! Furthermore, from the arguments presented in the introduction of this paper, the dependence on σ only occurs in the coupling of ϕ to \mathbf{T} and not in the helicity term of the functional. For finite element implementations, this means that the updated σ given by Eq. (22) only affects a relatively small part of one stiffness matrix, in each iteration!

III. HELICITY AND TIME DEPENDENCE

In this section we consider a time dependent version of the helicity functional and the relevance of metric independence in the solution of inverse problems. Let γ be an isotropic constitutive law, (\mathbf{F}, ϕ) a pair of complex functions, one vector and one scalar, and consider the functional

$$K(\mathbf{F}, \phi) = S(\mathbf{F}, \phi) + S^*(\mathbf{F}, \phi), \quad (23)$$

where

$$S(\mathbf{F}, \phi) = \int_{t_1}^{t_2} \int_V \mathbf{F}^* \cdot \left(\frac{\text{curl } \mathbf{F}}{2} + i \frac{\gamma}{2} \frac{\partial \mathbf{F}}{\partial t} + \gamma \text{ grad } \phi - i \frac{\partial \mathbf{P}}{\partial t} \right)$$

$$+ \frac{\phi^*}{2} \text{div } \mathbf{P} \, dV \, dt. \quad (24)$$

The Euler-Lagrange equation for $K(\mathbf{F}, \phi)$ is

$$\hat{D} \begin{pmatrix} \mathbf{F} \\ \phi \end{pmatrix} = \begin{pmatrix} \text{curl } \mathbf{F} - i \gamma \frac{\partial \mathbf{F}}{\partial t} + \gamma \text{ grad } \phi \\ -\text{div}(\gamma \mathbf{F}) \end{pmatrix} = \begin{pmatrix} i \frac{\partial \mathbf{P}}{\partial t} \\ -\text{div } \mathbf{P} \end{pmatrix}. \quad (25)$$

When $\gamma = 1$, we see that the operator \hat{D} defined by Eq. (25) is the operator \hat{D} of Eq. (10) if it were not for a missing time derivative of ϕ . This is an intentional omission since, if \mathbf{P} is real, Eq. (25) reduces to Maxwell's equations when

$$\mathbf{E} + i\mathbf{H} = \mathbf{F}, \quad (26)$$

$$\mathbf{D} + i\mathbf{B} = \gamma \mathbf{F}, \quad (27)$$

$$(\mathbf{J}, \rho) = \left(-\frac{\partial \mathbf{P}}{\partial t}, -\text{div } \mathbf{P} \right). \quad (28)$$

With these identifications the conservation of charge is satisfied automatically, the variables are assumed to be dimensionless, and we have the constitutive laws for chiral media if γ is complex. When ϕ is forced to be zero, Eqs. (25) are well known.^{7,11} However, for nonzero ϕ there are several novel aspects of the present formulation. First, ϕ is a Lagrange multiplier which imposes the divergence equations. That is Eq. (25) represents eight scalar equations which are Maxwell's equations in component form. Second, the operator \hat{D} represents a square root of the wave operator applied to \mathbf{F} and the square root of the Laplace operator applied to ϕ . Third, as in Sec. II, the square root operator has some nice metric independent aspects which can be exploited in the context of inverse problems. These follow from the appearance of the helicity term. The inverse problem connected to the complexified helicity functional is the inverse kinematic problem of seismology. See Sec. II of Ref. 10. In analogy to the real case, the metric independence of the helicity term ensures that only a small part of the functional needs to be updated when the constitutive law is updated.

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A topological invariant for the accessibility problem of micromagnetics

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A singularity-free three-dimensional micromagnetic configuration is accessible from a given initial state if there is a continuous family of interpolating states. This paper proposes an integral formula for the accessibility invariant which classifies (up to homotopy) the possible final states which are inaccessible from a given initial state. For infinite samples, we give explicit formulas describing initial states from which the final state of uniform saturation cannot be accessed.

I. INTRODUCTION

Fundamental to explaining the operation of micromagnetic memories is the assumption that information can be stored in topological configurations of magnetization which can be made energetically stable. The notion of a topological configuration is a consequence of the basic tenet of micromagnetics which asserts that the magnetization vector field is continuous and has a nonzero length. Unfortunately, there are circumstances where this basic tenet must be relaxed and one considers micromagnetic singularities. This paper introduces the notion of topological accessibility into micromagnetics, defines an accessibility invariant which distinguishes mutually inaccessible states, and relates the accessibility problem to micromagnetic singularities.

The previous investigation of the formal properties of Bloch points¹ plays an important role in the present paper for two reasons. First, for any two singularity-free configurations defined at times t_1 and t_2 , which are mutually inaccessible, one would like to have a concrete understanding of how and why Bloch points come into play. In the case where Bloch points cannot enter or exit through the boundary of a magnetic sample R , Bloch points must be created and annihilated in pairs of opposite "degree."¹ Hence, it is possible to regard the creation-evolution-annihilation cycle of a Bloch point pair as a closed curve in $R \times [t_1, t_2]$. For static configurations this helps us attribute inaccessibility to reverse magnetized tori which require the creation and annihilation of a Bloch point pair in order to collapse.

The second reason why this paper builds on a formal treatment of Bloch points is that the "accessibility invariant" is constructed from a "vector potential" for the solenoidal gyrovector field and the motivation for this construction is most transparent through the use of differential forms. Specifically, Slonczewski's gyrovector integral for the degree of a mapping into the Feldtkeller sphere,² when rephrased in terms of differential forms,¹ provides the language which relates the inaccessibility invariant, in the case of infinite samples, to the Hopf invariant.^{3,4}

Let us review the development of the Hopf invariant from both the differential form and gyrovector points of view. Recall¹ that a continuous, nowhere zero, three-dimensional, magnetization vector field \mathbf{m} can be normalized to have unit length and thought of as a mapping

$$f: R \rightarrow S^2, \quad (1)$$

where R is the magnetic sample and S^2 is the Feldtkeller sphere. If we introduce spherical coordinates (ϕ, θ) on S^2 then the normalized "volume form" on S^2 (element of area divided by total area) is

$$\omega = (1/4\pi) \sin \theta \, d\theta \wedge d\phi. \quad (2)$$

Definition (1) makes θ and ϕ functions of the coordinates on R . We can "pull back" ω via the mapping f to obtain a differential 2-form on R in terms of which we define the gyrovector, \bar{g} :

$$\begin{aligned} f^*\omega &= (1/4\pi) \bar{g} \cdot d\bar{S}, \\ \bar{g} &= \text{grad}(\cos \theta) \times \text{grad} \phi. \end{aligned} \quad (3)$$

Since pull backs and exterior differentiation commute ($f^*d = df^*$) and ω is a closed form on S^2 , we can conclude that the gyrovector is solenoidal. That is,

$$d\omega = 0 \Rightarrow 0 = d(f^*\omega) \Rightarrow \text{div} \bar{g} = 0. \quad (4)$$

Assume, for the time being that the magnetic sample R occupies all of \mathbb{R}^3 and is free of point defects. It is then natural to describe the gyrovector by a vector potential. That is, we would like to find a 1-form α or a \bar{G} such that

$$d\alpha = f^*\omega, \quad \text{i.e.,} \quad \text{curl} \bar{G} = \bar{g}. \quad (5)$$

Assuming that \mathbf{m} is sufficiently close to the constant vector field far away from the origin in \mathbb{R}^3 , the integral

$$I(f) = \int_{\mathbb{R}^3} \alpha \wedge f^*\omega = \frac{1}{(4\pi)^2} \int_{\mathbb{R}^3} \bar{G} \cdot \bar{g} \, dV \quad (6)$$

is finite and invariant under the "gauge transformation":

$$\alpha \rightarrow \alpha + d\chi, \quad \text{i.e.,} \quad \bar{G} \rightarrow \bar{G} + \text{grad} \chi, \quad (7)$$

provided that χ is close to constant far away from the origin.

The integral defined by Eq. (6) always takes on integer values. It is the purpose of this paper to show how the various nonzero integral values of (6) classify the micromagnetic configurations which are inaccessible from the field consisting of a constant \mathbf{m} throughout \mathbb{R}^3 and to generalize the construction to finite samples R and nonconstant initial or final states. Traditionally, the Hopf invariant is defined for mappings of S^3 (the unit sphere in \mathbb{R}^4) onto S^2 ,^{3,4} but, as we shall see, \mathbb{R}^3 is easily mapped onto S^3 by stereographic projection and the asymptotic conditions on \mathbf{m} (and hence, \bar{g}) ensure that f can be regarded as a map from S^3 to S^2 .

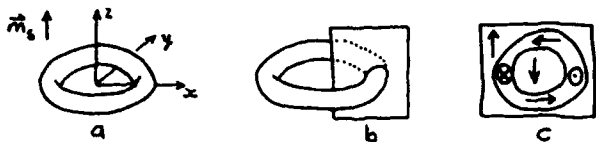


FIG. 1. A reverse magnetized torus.

II. THE HOPF INVARIANT AND MICROMAGNETICS

Given a magnetization configuration $\mathbf{m}(t, \mathbf{r})$ on all of \mathbb{R}^3 at some initial t_1 , we can ask whether there is a singularity-free time evolution such that at time t_2 , $\mathbf{m}(t_2, \mathbf{r})$ is some state of uniform saturation given by a constant vector \mathbf{m}_s . For example, consider the reverse magnetized torus of Fig. 1(a) where, in Feldtkeller's notation,⁵ Fig. 1(c) represents the cross section depicted in Fig. 1(b). A simple rescaling argument shows that shrinking the torus makes $\mathbf{m}(t, \mathbf{r})$ discontinuous in time at t_2 . Assuming that the torus would like to shorten its length without changing its overall diameter, we expect it to "break symmetry" and collapse by creating a Bloch point pair which is annihilated when the final state is achieved. In general, the integral of Eq. (6) gives an integer whose absolute value is, in some vague sense, the minimum number of Bloch point pairs required to be created and annihilated in order to obtain the state of saturation. In the case of Fig. 1 we shall see that

$$\frac{1}{16\pi^2} \int_{\mathbb{R}^3} \bar{\mathbf{G}} \cdot \mathbf{G} = 1. \quad (8)$$

In order to make this plausible, we need some formal results. Consider the unit sphere S^3 in \mathbb{R}^4 to be the unit sphere in \mathbb{C}^2 defined by $z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1$. Stereographic projection, denoted by σ_3 , then maps \mathbb{R}^3 onto S^3 where

$$\begin{aligned} \sigma_3: (x_1, x_2, x_3) &\rightarrow (z_1, z_2) \\ &= \frac{[2x_1 + 2ix_2, 2x_3 + i(1 - x_1^2 - x_2^2 - x_3^2)]}{1 + x_1^2 + x_2^2 + x_3^2}; \end{aligned} \quad (9)$$

$$\begin{aligned} \sigma_3^{-1}: (z_1, z_2) &\rightarrow (x_1, x_2, x_3) \\ &= \frac{(\text{Re } z_1, \text{Im } z_1, \text{Re } z_2)}{1 - \text{Im } z_2} \quad \text{when } \text{Im } z_2 \neq 1. \end{aligned} \quad (10)$$

Hence, when \mathbf{m} approaches the constant field as $x_1^2 + x_2^2 + x_3^2 \rightarrow \infty$ there is no loss of generality in considering the magnetization vector as a map $f\sigma_3^{-1}$ from S^3 to S^2 . In this case, Eq. (6) is the traditional Hopf invariant. The main results concerning the Hopf invariant are summarized by the following.^{3,4}

Theorem 1: (a) The Hopf invariant is independent of the choice of α (i.e., $\bar{\mathbf{G}}$) in Eq. (5). (b) Homotopic maps f have the same Hopf invariant. (c) If two maps, $f\sigma_3^{-1}$ and $\tilde{f}\sigma_3^{-1}$ have the same Hopf invariant then they are homotopic in S^3 .

Here homotopic means that there is a continuous family of interpolating states. For the above example of a reverse magnetized torus, the configuration is homotopic to the Hopf fibration for which a straightforward calculation³ shows that $I(f_1) = 1$. That is, Eq. (8) is indeed correct.

Frequently, the integral expression given by Eq. (6) is too cumbersome to be an effective computational tool. One

then falls back on the linking number interpretation of the Hopf invariant.³ Consider any two regular values p_1 and p_2 for the map f . That is, the Jacobian of f has rank 2 at p_1 and p_2 . The preimages of $f\sigma_3^{-1}$ of these two points are "compact manifolds" and, hence, disjoint circles in S^3 . The Hopf invariant is the linking number of these two circles:

$$I(f) = \text{Link}[f\sigma_3^{-1}(p_1), f\sigma_3^{-1}(p_2)]. \quad (11)$$

The proof of the equivalence of Eqs. (6) and (11) makes clear why the Hopf invariant is always an integer. Returning to Fig. 1 we see that $\text{Link}[f_1^{-1}(-\hat{i}_x), f_1^{-1}(+\hat{i}_y)]$ is equal to one. Hence, the reverse magnetized torus cannot evolve without singularities into the state of uniform saturation.

We now consider a more explicit treatment displaying states with higher Hopf invariant. A family of maps $f_{p,q}$ from \mathbb{R}^3 to S^2 describing a continuous nowhere vanishing magnetization field \mathbf{m} is given as a composition:^{4,6}

$$f_{p,q} = \sigma_2 h g_{p,q} \sigma_3, \quad (12)$$

where, introducing coordinates, we write

$$\mathbb{R}^3 \rightarrow S^3 \rightarrow S^3 \rightarrow \mathbb{CP}^1 \rightarrow S^2,$$

$$(x_1, x_2, x_3) \xrightarrow{\sigma_3} (z_1, z_2) \xrightarrow{g_{p,q}} (w_1, w_2) \xrightarrow{h} w \xrightarrow{\sigma_2} (m_x + im_y, m_z).$$

Here S^3 is the unit sphere in \mathbb{C}^2 , \mathbb{CP}^1 is the complex projective space, σ_3 and σ_2 are stereographic projection into three- and two-dimensional spheres, respectively, and $\sigma_2 h$ is the Hopf fibration. Explicitly, σ_3 is given by Eq. (9), while the remaining functions are defined as follows:

$$g_{p,q}: (z_1, z_2) \rightarrow (z_1^p, z_2^q) = (w_1, w_2) \quad \text{with } p, q \in \mathbb{Z};$$

$$h: (w_1, w_2) \rightarrow w_1/w_2 = w;$$

$$\sigma_2: w \rightarrow \frac{(2w, 1 - |w|^2)}{1 + |w|^2} = (m_x + im_y, m_z). \quad (13)$$

When $p = 1 = q$ the composite map is homotopic to the configuration described in Fig. 1 and from the discussion of Sec. II, we conclude that $I(\sigma_2 h) = 1$. Since the Hopf invariant can be described in terms of the degree of a map,³ it follows that

$$I(f_{p,q}\sigma_3^{-1}) = I(\sigma_2 h g_{p,q}) = \deg(g_{p,q}). \quad (14)$$

Also, since the function z^m has degree m when regarded as a map from the unit circle in the complex plane to itself, it follows from Eq. (14) that

$$I(f_{p,q}\sigma_3^{-1}) = \deg(g_{p,q}) = pq. \quad (15)$$

Thus from theorem 1 we conclude the following:

Theorem 2: (a) Two configurations $f_{p,q}$ and $f_{p',q'}$ defined by Eq. (12) are mutually inaccessible if $pq \neq p'q'$. (b) If $pq = p'q'$ then the maps $g_{p,q}$ and $g_{p',q'}$ are mutually accessible in S^3 but the map $f_{p,q}$ may not be accessible from $f_{p',q'}$ in \mathbb{R}^3 using a finite amount of energy.

To obtain a concrete and intuitive appreciation of these results, it is useful to obtain explicit formulas for the maps $f_{p,q}$ in terms of toroidal coordinates on \mathbb{R}^3 (Ref. 7):

$$\begin{aligned} (x_1 + ix_2, x_3) &= \frac{1}{\cosh(v) - \cos(u)} \\ &\times [\sinh(v)e^{i\phi}, \sin(u)], \end{aligned} \quad (16)$$

where $0 < v < \infty$, $0 < \phi < 2\pi$, $0 < u < 2\pi$. Substituting Eq. (16) into (9) and simplifying we obtain

$$(z_1, z_2) = \frac{1}{\cosh(v)} [\sinh(v)e^{i\phi}, e^{iu}]. \quad (17)$$

Equations (13) then give

$$(m_x + im_y, m_z) = \frac{(2\alpha e^{i(p\phi - qu)}, 1 - \alpha^2)}{1 + \alpha^2};$$

$$\alpha = \sinh^p(v) \cosh^{q-p}(v). \quad (18)$$

Equation (18) describes, in terms of toroidal coordinates, continuous, nowhere zero magnetization distributions with Hopf invariant pq . The special case of $p = q = 1$ is described in Fig. 1. In general there is a $2\pi p$ twist in the direction of increasing ϕ and a $2\pi q$ twist in the direction of increasing u . Returning to the linking number interpretation of the Hopf invariant, the generic curves of constant \mathbf{m} , with linking number pq , are found by fixing $p\phi - qu$ and v .

III. THE ACCESSIBILITY INVARIANT

We now generalize the Hopf invariant to obtain a topological accessibility invariant which is useful in the context of finite samples for which the magnetization is given throughout the sample at initial and final times and on the boundary of the sample as a function of time. Consider a finite region R and a time interval $[t_1, t_2]$ denoted by I . The region R is assumed to have no defects or "holes" so that the (co)homology groups are trivial. The magnetization on the boundary $\partial(R \times I)$ of the region of space-time is fixed by specifying $\mathbf{m}(t_1, \mathbf{r})$ and $\mathbf{m}(t_2, \mathbf{r})$ for all $\mathbf{r} \in R$ and $\mathbf{m}(t, \mathbf{r})$ for $t_1 \leq t \leq t_2$ and $\mathbf{r} \in \partial R$. Here it is natural to regard time as the parameter describing the homotopy between mutually accessible initial and final states. The situation is as before except that a continuous, nowhere zero magnetization field, if it exists, would be described by a map \tilde{f} which extends f :

$$\tilde{f}: R \times I \rightarrow S^2, \text{ where } f: \partial(R \times I) \rightarrow S^2. \quad (19)$$

If R has trivial (co)homology groups then it is possible to find a potential 1-forms α and $\tilde{\alpha}$ to describe the pull back of the normalized volume form on S^2 :

$$d\tilde{\alpha} = \tilde{f}^*\omega \text{ on } R \times I, \quad d\alpha = f^*\omega \text{ on } \partial(R \times I), \quad (20)$$

Consider the integral defining the accessibility invariant:

$$J(f) = \int_{\partial(R \times I)} \alpha \wedge d\alpha. \quad (21)$$

The value of this integral is zero if the map f can be extended to a singularity-free map \tilde{f} because if such an \tilde{f} exists, by Stokes' theorem and Eq. (20) we have

$$\begin{aligned} J(f) &= J(\tilde{f}) = \int_{\partial(R \times I)} \tilde{\alpha} \wedge d\tilde{\alpha} = \int_{R \times I} d(\tilde{\alpha} \wedge d\tilde{\alpha}) \\ &= \int_{R \times I} d\tilde{\alpha} \wedge d\tilde{\alpha} = \int_{R \times I} [\tilde{f}^*(\omega) \wedge \tilde{f}^*(\omega)] \\ &= \int_{R \times I} \tilde{f}^*(\omega \wedge \omega) = 0 \text{ since } \omega \wedge \omega = 0 \text{ on } S^2. \end{aligned}$$

Hence a nonzero value for $J(f)$ expresses the impossibility of extending the data given on $\partial(R \times I)$ to a continuous nonzero distribution of magnetization on $R \times I$. As in the case of the Hopf invariant, Stokes' theorem is used to verify that the

integral $J(f)$ is invariant under the gauge transformation $\alpha \rightarrow \alpha + d\chi$. The linking number interpretation is also valid so that the integral $J(f)$ takes on integral values which represent the linking number of two preimages of f in the closed manifold $\partial(R \times I)$. Indeed we expect such a result since Eq. (21) reduces to Eq. (6), when \mathbf{m} is a constant vector on ∂R for all time, over R at t_2 , and on $R^3 - R$ at t_1 .

A concrete understanding of Eq. (21) in terms of vector and scalar functions on $R^3 \times I$ is obtained by appealing to an analogy with electrodynamics. The electromagnetic field is described by a differential 2-form β on space-time which being closed can be expressed as the exterior derivative of a potential 1-form α :

$$d\alpha = \beta = \bar{\mathbf{B}} \cdot d\bar{\mathbf{S}} + \bar{\mathbf{E}} \cdot d\bar{\mathbf{l}} dt,$$

where

$$\alpha = \bar{\mathbf{A}} \cdot d\bar{\mathbf{l}} - \phi dt. \quad (22)$$

In micromagnetics, β plays the role of $f^*\omega$ so that the vector fields $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are formally analogous to $\bar{\mathbf{G}}$ and $\bar{\mathbf{g}}$. Thus, to complete the prescription given by Eq. (20) we need to introduce an analog of ϕ and \mathbf{E} . Introducing a scalar potential γ , the equation $d\alpha = \beta$ shows that Eq. (20) can be rewritten as

$$d\alpha = \text{curl } \bar{\mathbf{G}} \cdot d\bar{\mathbf{S}} + \left(\text{grad } \gamma - \frac{\partial \bar{\mathbf{G}}}{\partial t} \right) \cdot d\bar{\mathbf{l}} \wedge dt = f^*\omega. \quad (23)$$

Once a potential $(\bar{\mathbf{G}}, \gamma)$ is found Eq. (21) becomes

$$\begin{aligned} J(f) &= \left\{ \int_{t_1}^{t_2} \int_{\partial R} \left[\bar{\mathbf{G}} \times \left(\text{grad } \gamma - \frac{\partial \bar{\mathbf{G}}}{\partial t} \right) \cdot d\bar{\mathbf{l}} - \gamma \text{curl } \bar{\mathbf{G}} \right] d\bar{\mathbf{S}} dt \right. \\ &\quad \left. + \int_R \bar{\mathbf{G}} \cdot \text{curl } \bar{\mathbf{G}}|_{t_2} dV - \int_R \bar{\mathbf{G}} \cdot \text{curl } \bar{\mathbf{G}}|_{t_1} dV \right\} \frac{1}{16\pi^2} \end{aligned} \quad (24)$$

Invariance of Eqs. (23) and (24) under gauge transformations of the form

$$\mathbf{G} \rightarrow \mathbf{G} + \text{grad } \chi, \quad \gamma \rightarrow \gamma + \frac{\partial \chi}{\partial t} \quad (25)$$

are still best understood in the context of equation " $\alpha \rightarrow \alpha + d\chi$ ". The results of theorem 1 and the calculation below Eq. (21) are stated as follows:

Theorem 3: The integral given in (21) or (24) takes on integer values which are nonzero only if the initial and final states given in (19) are mutually inaccessible. If R is a contractible region of finite extent, the states are topologically accessible whenever the integral has a zero value.

The relevance of Eq. (6) to micromagnetics has also been noted by P. Asselin, who provided the example illustrated in Fig. 1. The authors would like to thank Honeywell Inc. for financial support of this work and Floyd Humphrey for his encouragement.

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